HEAD-TAIL MODES FOR STRONG SPACE CHARGE

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Abstract

Head-tail modes are described here for the space charge tune shift significantly exceeding the synchrotron tune. General equation for the modes is derived. Spatial shapes of the modes, their frequencies, and coherent growth rates are explored. The Landau damping rates are also found. Suppression of the transverse mode coupling instability by the space charge is explained.

INTRODUCTION

The head-tail instability of bunched beams was observed and theoretically described many years ago [1-3]. Since then, this explanation has been accepted and included in textbooks [4,5], but still there is an important gap in the theory of head-tail interaction. This relates to an influence of space charge on the coherent modes: their shapes, growth rates and Landau damping. A significant theoretical paper on that issue was published by M. Blaskiewicz ten years ago [6]. In particular, a compact analytical description of the coherent modes was found there for a square well model, air-bag distribution and a short-range wake function, without any assumption for the relative values of the space charge tune shift, the synchrotron tune and the coherent tune shift. Ref. [6] shed a first light on the problem, reminding that beyond very specific restrictions of its model, all the questions had yet to be answered. Here, an attempt to provide an answer is presented. Compared to Ref. [6], this attempt is both broader and narrower. It is broader since there are no assumptions here about the potential well and the bunch distribution functions; the suggested method is applicable for any potential well or distribution function. The solution for a parabolic potential well and 3D Gaussian bunch is given in details, but the method is universal. Since this paper deals with arbitrary distribution functions, the Landau damping is not generally zero here and it can be and is calculated in this paper. From another aspect though, my approach here is narrower than that of Ref. [6], since a certain condition between the important parameters is assumed below. Namely, it is assumed that the space charge tune shift in the bunch 3D center $Q_{\max}$ is large compared to both to the synchrotron tune Q_s and the wake-driven coherent tune shift Q_w: $Q_{\max} >> Q_s, Q_w$.

The organization of this paper is as follows. In the next chapter, a single-particle equation of motion is written in the rigid-beam approximation. The validity of this approximation is discussed. Then, the coherent modes are found from this equation for the square potential well with arbitrary beam distribution function; vanishing Landau damping for the square well is pointed out.

After that, the problem for the coherent modes in the presence of strong space charge is reduced to a second-order ordinary differential equation with zero boundary conditions. The eigen-modes and eigen-frequencies are found for no-wake case, when the space charge and the synchrotron motion only are taken into account. At this step, the mode shapes $\bar{y}_k(\tau)$ and frequencies ν_k are found and the wake-driven coherent growth rates are calculated as perturbations. When the mode structure was described both in general and in details for the Gaussian bunch, the Landau damping rates Λ_k were found in in the next two chapters: first, without and second, with transverse nonlinearity. In the last section of this paper, the limit of weak head-tail is removed, and the transverse mode coupling instability (TMCI) at strong space charge is discussed. It is shown that in this case, the TMCI threshold typically exceeds the separation of the nearest modes by a factor of 10-100.

RIGID BEAM EQUATIONS

Using the same notations as in Ref. [6], let θ be time in radians, τ be a distance along the bunch in radians as well, $X_i(\theta)$ be a betatron offset of i-th particle, and $\bar{X}(\theta,\tau)$ be an offset of the beam center at the given time θ and position τ. Since all the tune shifts are small compared with the bare betatron tune Q_b, the latter can be excluded from the considerations by using slow variables $x_i(\theta)$:

$$X_i(\theta) = \exp(-iQ_b\theta)x_i(\theta).$$

After that, a single-particle equation of motion can be written as

$$\dot{x}_i(\theta) = iQ(\tau_i(\theta))\left[x_i(\theta) - \bar{x}(\theta,\tau_i(\theta))\right] - i\zeta \mathrm{v}_i(\theta)x_i(\theta) - i\kappa\hat{\mathbf{W}}\bar{x}\,. \quad (1)$$

Here a dot stands for a time derivative, $\dot{x}_i = dx_i/d\theta$; the effective chromaticity $\zeta = -\xi/\eta$ with $\xi = dQ_b/d(\Delta p/p)$ as the conventional chromaticity and $\eta = \gamma_t^{-2} - \gamma^{-2}$ as the slippage factor; $Q(\tau)$ is the space charge tune shift as a function of the position inside the bunch; $\mathrm{v}_i(\theta) = \dot{\tau}_i(\theta)$ is the velocity within the bunch, and $\kappa\hat{\mathbf{W}}\bar{x}$ is the wake force expressed in terms of the wake linear operator $\hat{\mathbf{W}}$ to be specified below. Note that Eq. (1) already assumes a rigid-beam approximation. This approximation is based on the idea that the transverse coherent motion of the beam can be treated as displacements of beam

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longitudinal slices, so the force on a given particle is just proportional to its offset from the local beam centroid. For a coasting beam, validity of the rigid-beam model is discussed in Ref. [7]. To be justified, the rigid-beam model requires a sufficient separation between the coherent frequency and the incoherent spectrum: the separation has to be significantly larger than the width of the bare incoherent spectrum. As a result almost all the particles respond almost identically to the collective field.

The chromaticity term can be excluded from Eq. (1) with a substitution $x_i(\theta)=y_i(\theta)\exp(-i\zeta\tau_i(\theta))$, leading to

$$\dot{y}_i(\theta)=iQ(\tau_i(\theta))\left[y_i(\theta)-\bar{y}(\theta,\tau_i(\theta))\right]-i\kappa\hat{\mathbf{W}}\bar{y} \tag{2}$$

with

$$\kappa=\frac{r_0 R}{4\pi\beta^2\gamma Q_b};$$

$$\hat{\mathbf{W}}\bar{y}=\int_{\tau}^{\infty}W(\tau-s)\exp(i\zeta(\tau-s))\rho(s)\bar{y}(s)ds. \tag{3}$$

Here r_0 is the classical radius of the beam particles; $R=C/(2\pi)$ is the average ring radius; β and γ are the relativistic factors, $\rho(s)$ is the bunch linear density normalized on the number of particles in the bunch, $\int\rho(s)ds=N_b$, and the wake-function $W(s)$ is defined according to Ref [4] (slightly different from the definition of Ref. [6]).

First, Eq. (2) is solved below for no-wake case. After that, the wake is taken into account as a perturbation of the space charge eigen-modes. These unperturbed eigen-modes are to be found from a no-wake reduction of Eq. (2):

$$\dot{y}_i(\theta)=iQ(\tau_i(\theta))\left[y_i(\theta)-\bar{y}(\theta,\tau_i(\theta))\right] \tag{4}$$

Solutions of this equation give the space charge eigen-modes: their spatial shapes, frequencies and Landau damping rates. A detailed analysis of that is given below. Here though, an important feature of the zero-wake spectrum can be already pointed out: the damping rates of the space charge modes do not depend on the chromaticity at all. The shapes of the modes do not depend on the chromaticities as well, except for the common head-tail phase factor $\exp(-i\zeta\tau)$. The chromaticity enters into the problem through the wake term only, Eq. (3), affecting the coherent growth rates. As it will be seen below, the chromaticity normally makes the coherent growth rates negative for the modes, which number k is smaller than the head-tail phase, $k\,\square\,\zeta\sigma$, with σ as the rms bunch length. When the wake-driven tune shift is comparable with a distance between the nearest modes, the chromaticity affects not only mode shapes, but the Landau damping as well.

SQUARE POTENTIAL WELL

Before going to a general case, it would be instructive to solve easier problem for the square potential well. In contrast to Ref. [6], where the air-bag distribution is assumed, here this problem is solved for a general distribution over the synchrotron frequencies, but within the weak head-tail constraint $Q_w \ll \min\left(Q_s^2/Q_{\max},Q_s\right)$. In this chapter only, the ratio between the space charge tune shift and the synchrotron tune can be arbitrary. Following the described procedure, the first step is to find the modes for zero wake field. The wake-driven coherent shifts are calculated in the next step as perturbations.

For no-wake case, a single-particle equation (4) has a constant coefficient $Q(\tau)=Q$, and can be easily solved:

$$y_i(\theta)=-iQ\int_{-\infty}^{\theta}\bar{y}(\theta',\tau_i(\theta'))\exp(iQ(\theta-\theta'))d\theta'\cdot \tag{5}$$

To find the eigen-modes, the boundary conditions for the beam centroid have to be taken into account. Since every particle is reflected instantaneously from the walls of the potential well, its offset derivative cannot immediately change after the reflection. This, in turn, leads to a conclusion that a space derivative of the beam centroid is zero at the bunch boundaries $\tau=0$ and $\tau=l$:

$$\left.\frac{\partial}{\partial\tau}\bar{y}(\theta,\tau)\right|_{\tau=0}=\left.\frac{\partial}{\partial\tau}\bar{y}(\theta,\tau)\right|_{\tau=l}=0\cdot \tag{6}$$

Thus, the centroid offset can be Fourier-expanded as

$$\bar{y}(\theta,\tau)=\exp(-i\nu_k\theta)\sum_{m=0}^{\infty}C_m\cos(\pi m\tau/l), \tag{7}$$

with C_m as yet unknown coefficients, and ν_k as the eigen-frequencies to be found. For the right-hand side of single-particle Eq. (5), it gives

$$\bar{y}(\theta',\tau_i(\theta'))=\exp(-i\nu_k\theta')\sum_{m=0}^{\infty}C_m\cos\left(\pi m\frac{\tau_i(\theta)-\mathrm{v}_i(\theta-\theta')}{l}\right). \tag{8}$$

Being substituted into Eq. (5) and after taking the integral, this results in

$$y_i(\theta)=\exp(-i\nu_k\theta)\sum_{m=0}^{\infty}C_m\cos\left(\pi m\frac{\tau_i(\theta)}{l}\right)\frac{Q(Q+\nu_k)}{(Q+\nu_k)^2-m^2Q_{si}^2}, \tag{9}$$

with $Q_{si}=\pi\mathrm{v}_i/l$ as the synchrotron frequency of the i-th particle. Averaging Eq. (9) over all the particles at the given location yields the shape of the eigen-modes as

$$\bar{y}_k(\theta,\tau)=\exp(-i\nu_k\theta)\bar{y}_k(\tau);\ \bar{y}_k(\tau)=\sqrt{2/l}\cos(\pi k\tau/l). \tag{10}$$

These eigen-modes constitute a full orthonormal basis:

$$\int_0^l\bar{y}_k(\tau)\bar{y}_m(\tau)d\tau=\delta_{km}\cdot \tag{11}$$

The coherent shifts ν_k have to be found from the following dispersion equation:

$$1=\int\frac{Q(Q+\nu_k)f(Q_s)dQ_s}{(Q+\nu_k+i0)^2-k^2Q_s^2};\ k=0,1,2...\cdot \tag{12}$$

Here, the Landau rule $\nu_k\to\nu_k+i0$ was taken into account, and the distribution function over the synchrotron frequencies is assumed to be normalized:

$$\int f(Q_s)dQ_s=1\ \cdot$$

A simplest case for the dispersion equation (12) is the air-bag distribution, fully considered in Ref. [6]. Taking $f(Q_s)=\delta(Q_s-\bar{Q}_s)$, the result of Ref. [6] is reproduced:

$$\nu_k = -\frac{Q}{2} \pm \sqrt{\frac{Q^2}{4} + k^2 \bar{Q}_s^2} \qquad (13)$$

Note that the positive and negative frequencies are significantly different. If the synchrotron tune is so small that $k\bar{Q}_s << Q/2$, the negative eigen-frequencies almost coincide with the singe-particle tune shift, while the positive ones are much smaller than that. The negative solution $\nu_k \approx -Q$ is almost equal to the single-particle tunes, so it likely does not satisfy the condition for the rigid-beam approximation, it is easily Landau-damped, and thus is excluded from the further analysis. The other solution yields in this case

$$\nu_k = k^2 \bar{Q}_s^2 / Q \quad . \qquad (14)$$

Note that instead of zero space charge spectrum $Q_k = k\bar{Q}_s$, the mode frequency here is quadratic with its mode number.

When there is some synchrotron frequency spread, it is possible for some particles to be resonant with the mode, providing the Landau damping. For them, the denominator in the dispersion equation goes to zero, so their synchrotron tunes are

$$kQ_s = \pm(Q + \nu_k) \approx \pm Q \ . \qquad (15)$$

For typical practical cases, where the synchrotron tune is order(s) of magnitude smaller, than the maximal space charge tune shift, this condition selects very distant tails, so we conclude that as a practical matter there is no Landau damping here. Up to this point, the nonlinear betatron tune shift was not taken into account. This tune shift $\delta Q(J_1, J_2)$, being a function of the two transverse actions J_1, J_2, modifies the dispersion relation similarly to the coasting beam case, resulting in

$$1 = -\int \frac{\partial f}{\partial J_1} \frac{J_1 Q(Q+\nu_k) dQ_s dJ_1 dJ_2}{(Q - \delta Q(J_1, J_2) + \nu_k + i0)^2 - (kQ_s)^2}, \qquad (16)$$

with the normalized distribution function $\int f(Q_s, J_1, J_2) dQ_s dJ_1 dJ_2 = 1$ assuming studying oscillations along the first degree of freedom. The dispersion relation (16) is valid for any dependence of the space charge tune shift on the transverse actions $Q \to Q(J_1, J_2)$. When the space charge tune shift is much larger than the synchrotron tune and the nonlinear tune shift, the solution of the dispersion equation (16) follows:

$$\nu_k = k^2 \frac{\int \frac{f Q_s^2 dQ_s dJ_1 dJ_2}{Q^2(J_1, J_2)}}{\int \frac{f dQ_s dJ_1 dJ_2}{Q(J_1, J_2)}} + 2 \frac{\int \frac{f \delta Q(J_1, J_2) dQ_s dJ_1 dJ_2}{Q(J_1, J_2)}}{\int \frac{f dQ_s dJ_1 dJ_2}{Q(J_1, J_2)}}$$

Since the obtained solution does not assume any special relation between the space charge tune shift and the synchrotron tune, it is worthwhile to look at the case of relatively low space charge, $Q_s >> Q/2$. As it is clear from Eq. (13), the collective modes are separated from the incoherent spectrum by one-half of the space charge tune shift. Without transverse nonlinearity, the Landau damping is provided by the synchrotron tune spread. Namely, for k-th mode, it is provided by particles whose synchrotron tune Q_s deviates from the average synchrotron tune $\bar{Q}_s$ by $Q_s - \bar{Q}_s = -Q/(2k)$. For $k = 0$, there is no Landau damping without transverse nonlinearity. Note that for this low space charge case, the rigid-beam approximation is valid for both signs in Eq. (13), as soon as all the resonant particles are located only in tails of the distribution.

After the solution (10), (16) is obtained, it is worth to note that a detailed study of the Landau damping for the square well model does not appear to make much sense, since this model greatly underestimates the Landau damping. The reason is that in case of the square well, there is no spatial variation of the space charge tune shift. However, for realistic buckets and bunch shapes, the space charge tune shift smoothly drops to zero at the longitudinal tails, making possible the wave-to-particle energy transfer there. This appears to be a leading mechanism of Landau damping for bunched beams, missing in the square well case, and being considered in detail in two special chapters below.

To finish our analysis of the square well model, there is one more thing to do.. After the set of the eigen-modes is found for zero-wake case, the wake can be taken into account by means of perturbation theory, assuming it is small enough. This step is relatively simple. Indeed, the wake term in Eq. (2) causes its own tune shift Q_w, leading to additional factor $\exp(-iQ_w\theta)$ for the single-particle offset in the left-hand side of Eq. (2). This immediately turns this equation into $Q_w y_i = \hat{\mathbf{W}}\bar{y}$. After averaging, this gives the wake tune shift as a diagonal element of the wake operator:

$$Q_w = (\bar{y}, \hat{\mathbf{W}}\bar{y}) \equiv \kappa \int_{-\infty}^{\infty}\int_{\tau}^{\infty} W(\tau - s) \exp(i\zeta(\tau - s)) \rho(s) \bar{y}_k(s) \bar{y}_k(\tau) ds d\tau. \quad (17)$$

Here, the normalization (11) of the orthogonal modes $\bar{y}_k(\tau)$ (Eq. 10) was taken into account.

If the vacuum chamber is not round, the detuning, or quadrupole wake $D(\tau)$ modifies the coherent tune shifts [8]. Assuming a force from a leading particle (subscript 1) on a trailing particle (subscript 2) as $\propto W(\tau)x_1 + D(\tau)x_2$, it yields for the detuning coherent shift

$$Q_d = \kappa \int_{-\infty}^{\infty}\int_{\tau}^{\infty} D(\tau - s) \rho(s) \bar{y}_k^2(\tau) ds d\tau \, . \qquad (18)$$

Instead of the wake tune shift (17), the detuning one (18) is purely real, it does not affect the beam stability. This conclusion though is limited by the weak head-tail approximation, where the wake is so small that it can be taken as perturbation, leading to (17, 18). It was shown in Ref. [8], that it is not valid for the transverse mode coupling instability (TMCI), where the detuning wake normally increases the intensity threshold.

Note, that the derivation of Eqs. (17, 18) does not use any specific features of the square well model; thus, these results are valid for any potential well and bunch profile, as soon as a corresponding orthonormal basis of the eigen-modes is used.

Growth rates as functions of the head-tail phase ζl are presented in Fig. 1 for the square well model with a constant wake function, $W(\tau)= W_0$ =const. The rates are given in units of $\kappa N_b W_0$ with $N_b = \rho l$ as a number of particles in the bunch.

GENERAL SPACE CHARGE MODES

In this chapter, an ordinary differential equation for the eigen-modes is derived for a general potential well and 3D bunch distribution function, assuming strong space charge,

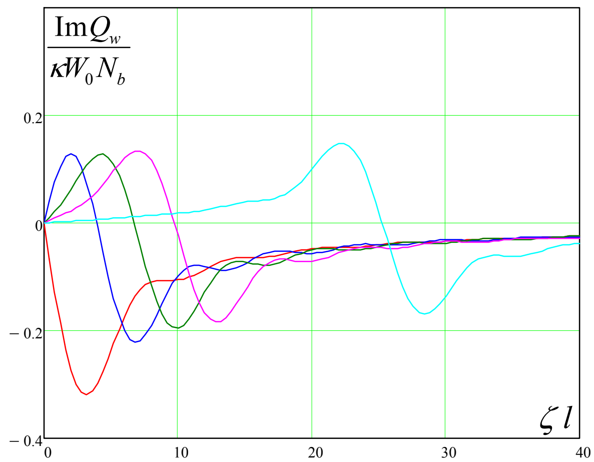

Fig. 1: Growth rates of the lowest mode (mode 0, red), mode 1 (blue), 2 (green), 3 (magenta) and 8 (cyan) as functions of the head-tail phase ζl for a constant wake function. The rates are in units of $\kappa N_b W_0$.

$$Q >> 2k\bar{Q}_s \, . \tag{19}$$

The fact that the bunch modes are described by single-argument functions, dependent on the position along the bunch only, is due to a strong coherence, introduced by the space charge. Indeed, the classical no-space-charge head-tail modes are generally described by their dependence both on the synchrotron phase and the synchrotron action, so the synchrobetatron modes are characterized by what are called as azimuthal and radial numbers; all the radial modes are degenerated?, having the same coherent tune, determined by the azimuthal number. With the strong space charge, all the individual degrees of freedom are detuned from the coherent motion by approximately the same number, namely, the local space charge tune shift. So, locally all the particles are moving almost identically; their position in the synchrotron phase space does not play a role, as soon as they are at the given longitudinal position. That is why the space charge modes are described by single-argument functions dependent on the position along the bunch only.

The single-particle equation (4) can be solved in general:

$$y_i(\theta) = -i\int_{-\infty}^{\theta} Q(\tau_i(\theta'))\bar{y}(\theta', \tau_i(\theta'))\exp(i\Psi(\theta) - i\Psi(\theta'))d\theta' \, ; \tag{20}$$
$$\Psi(\theta) \equiv \int_0^{\theta} Q(\tau_i(\theta'))d\theta' \quad .$$

Since $Q(\tau) > 0$, dependence $\Psi(\theta)$ is monotonic and so integration over θ in Eq. (20) can be replaced by integration over Ψ:

$$y_i(\Psi) = -i\int_{-\infty}^{\Psi} \bar{y}(\Psi')\exp(i\Psi - i\Psi')d\Psi' \, . \tag{21}$$

Note that due to (19), the phase Ψ runs fast compared with relatively slow dependence $\bar{y}(\Psi)$, so the later can be expanded in a Taylor series-,

$$\bar{y}(\Psi') \approx \bar{y}(\Psi) - (\Psi - \Psi')\frac{d\bar{y}}{d\Psi} + \frac{(\Psi - \Psi')^2}{2}\frac{d^2\bar{y}}{d\Psi^2} \, .$$

After that the integral is easily evaluated:

$$y_i(\Psi) = \bar{y}(\Psi) - i\frac{d\bar{y}}{d\Psi} - \frac{d^2\bar{y}}{d\Psi^2} \, . \tag{22}$$

To come back to original variables, it can be used that

$$\frac{d}{d\Psi} = \frac{\mathrm{v}}{Q(\tau)}\frac{\partial}{\partial\tau} + \frac{1}{Q(\tau)}\frac{\partial}{\partial\theta} = \frac{1}{Q(\tau)}\left(\mathrm{v}\frac{\partial}{\partial\tau} - i\nu_k\right). \tag{23}$$

Applied to Eq. (22), this gives

$$y_i = \bar{y} - \frac{i}{Q(\tau)}\left(\mathrm{v}\frac{\partial}{\partial\tau} - i\nu_k\right)\bar{y} - \left[\frac{1}{Q(\tau)}\left(\mathrm{v}\frac{\partial}{\partial\tau} - i\nu_k\right)\right]^2 \bar{y} \, . \tag{24}$$

At this point, we can average over velocities v at the given position τ. Doing this, the eigenvalue ν_k can be neglected in the second-order term of Eq. (24), supposing $|\nu_k| << |\mathrm{v}\partial/\partial\tau| \cong kQ_s$, as it is true for the square well bucket and is confirmed below for general case. After that, the equation for eigen-modes follows as a second-order ordinary self-adjoint differential equation:

$$\nu_k\bar{y} + u(\tau)\frac{d}{d\tau}\left(\frac{u(\tau)}{Q(\tau)}\frac{d\bar{y}}{d\tau}\right) = 0; \tag{25}$$
$$u(\tau) \equiv \sqrt{\int_{-\infty}^{\infty}\mathrm{v}^2 f(\mathrm{v}, \tau)d\mathrm{v}} \, ,$$

where $f(\mathrm{v},\tau)$ is a normalized steady-state longitudinal distribution function, $f(\mathrm{v},\tau) = f(H(\mathrm{v},\tau))$, with $H(\mathrm{v},\tau)$ as the longitudinal Hamiltonian.

It has to be noted, that the derivation of Eq. (25) from Eq. (24) implicitly assumed that the space charge tune shift depends only on the longitudinal position, and does not depend on the individual transverse amplitude; in other words, the KV distribution was assumed. It is possible, however, to remove this limitation, and to see that Eq. (25) is actually valid for any transverse distribution, after certain redefinition of the space charge tune shift $Q(\tau)$. Indeed, single-particle Eq. (24) does not make any assumption about the individual space charge tune shift dependence $Q(\tau)$, which can be considered as dependent on the transverse actions J_{1i}, J_{2i} as well: $Q(\tau) \to Q_i(\tau) = Q(J_{1i}, J_{2i}, \tau)$. After that, averaging of Eq.(24) just has to take into account this dependence of the space charge tune shift on the transverse actions. As an example, for a Gaussian round beam, i. e. a beam with identical emittances and beta-functions, the transverse dependence of the space charge tune shift can be calculated as (see e. g. Ref [5]):

$$Q(J_1,J_2,\tau)=Q_{\max}(\tau)\int_0^1 \frac{\left[\mathrm{I}_0\left(\frac{J_1 z}{2}\right)-\mathrm{I}_1\left(\frac{J_1 z}{2}\right)\right]\mathrm{I}_0\left(\frac{J_2 z}{2}\right)}{\exp\left(z(J_1+J_2)/2\right)}dz \equiv \tag{26}$$

$$\equiv Q_{\max}(\tau)g(J_1,J_2)$$

Here J_1, J_2 are two dimensionless transverse actions, conventionally related to the offsets as $x=\sqrt{2J_1\varepsilon_1\beta_1}\cos(\psi)$ with ε_1 and β_1 as the rms emittance and beta-function, so that the transverse distribution function looks as

$$f_\perp(J_1,J_2)=\exp(-J_1-J_2)\,. \tag{26a}$$

The transverse averaging of Eq. (24) requires calculation of two transverse moments q_{-1}, q_{-2} of the tune shift $Q(J_1,J_2,\tau)$ generally defined by:

$$\left\langle Q_i^p(\tau)\right\rangle_\perp = \int_0^\infty\int_0^\infty dJ_1 dJ_2 f_\perp(J_1,J_2) Q^p(J_1,J_2,\tau) \equiv q_p^p Q_{\max}^p(\tau)\,.$$

$$q_p = \left[\int_0^\infty\int_0^\infty dJ_1 dJ_2 f_\perp(J_1,J_2) g^p(J_1,J_2)\right]^{1/p} \tag{27}$$

After that, Eq. (25) follows for any transverse distribution with a substitution

$$Q(\tau)\to Q_{\mathrm{eff}}(\tau)\equiv\left(q_{-2}^2/q_{-1}\right)Q_{\max}(\tau)$$

For the round Gaussian distribution, Eq. (26, 26a), $q_{-1}=0.58$, $q_{-2}=0.55$, $q_{-3}=0.52$, yielding $q_{-2}^2/q_{-1}=0.52$.

Thus, Eq. (25) for eigenvalues ν_k and eigenfunctions $\bar{y}$ is valid for arbitrary beam transverse distribution, shape of the longitudinal potential well and arbitrary longitudinal distribution *f*(*H*). Even if the longitudinal and transverse distributions are not factorized, Eq. (25) is still valid after proper modifications of the functions $u(\tau)$ and $Q(\tau)$.

A question of boundary conditions for this equation is a bit subtle. What happens at $t\to\pm\infty$? May the eigenfunctions tend to an arbitrary finite constant there? The answer proves to be negative: the eigenfunctions must be zero at infinity. When a particle goes from the bunch longitudinal center to its tail, the space charge tune shift goes down to zero. Far enough from the center, the space charge tune shift gets to be as small as the synchrotron tune. At these high longitudinal offsets, all the assumptions of the model used here are violated. First, the rigid-beam model is not correct there, and, second, the strong space charge condition (19) is not valid. At these distances, the coherent frequency stays within the local incoherent spectrum, resulting in the fast decoherence of the collective motion beyond this point. In other words, the coherent signal cannot pass through a point where the space charge tune shift and the synchrotron tune are comparable; the eigen-mode goes to zero after that. Thus, the self-adjoint boundary problem (25) has to be solved with zero boundary conditions:

$$\bar{y}(\pm\infty)=0\,. \tag{28}$$

Eqs (25, 28) reduce the general problem of eigen-modes to a well-known mathematical boundary-value problem, similar to the single-dimensional Schrödinger equation (see e. g. [9]). This problem is normal, so it has full orthonormal basis of the eigen-functions

$$\int_{-\infty}^{\infty}\bar{y}_k(\tau)\bar{y}_m(\tau)\frac{d\tau}{u(\tau)}=\delta_{km}\,. \tag{29}$$

As a consequence,

$$\sum_{m=0}^{\infty}\bar{y}_m(\tau)\bar{y}_m(s)=u(\tau)\delta(s-\tau)\,. \tag{30}$$

At the bunch core, the *k*-th eigen-function $\bar{y}_k(t)$ behaves like $\sim\sin(k\tau/\sigma)$ or $\sim\cos(k\tau/\sigma)$, and the eigenvalues are estimated to be

$$\nu_k \cong k^2\overline{Q}_s^2/Q_{\mathrm{eff}}(0) << \overline{Q}_s\ , \tag{31}$$

which are similar to the values in the square well case.

Due to the general modification of the orthogonality by Eq. (29) compared to the square well case of Eq. (11), the formulas for the coherent tune shift and the coherent detuning, Eqs (17,18) have to be generally modified by a substitution $d\tau\to d\tau/u(\tau)$:

$$Q_w=\kappa\int_{-\infty}^{\infty}\int_{\tau}^{\infty}W(\tau-s)\exp(i\zeta(\tau-s))\rho(s)\bar{y}_k(s)\bar{y}_k(\tau)u^{-1}(\tau)ds d\tau\,,$$

$$Q_d=\kappa\int_{-\infty}^{\infty}\int_{\tau}^{\infty}D(\tau-s)\rho(s)\bar{y}_k^2(\tau)u^{-1}(\tau)ds d\tau\,. \tag{32}$$

Using Eq. (30), a sum of all growth rates can be proven to be zero:

$$\sum_{k=0}^{\infty}\mathrm{Im}Q_w=0\,. \tag{33}$$

This statement sometimes is referred to as the growth rates sum theorem. Note also, that the detuning wake does not introduce any growth rate, and every growth rate is proportional to the head-tail phase when this phase is small, similar to the conventional no-space-charge case.

For a short wake, $W(\tau)=-G\delta'(\tau)$, the growth rate can be expressed as

$$\mathrm{Im}Q_w=\kappa\rho_k\zeta G;\quad \rho_k\equiv\int_{-\infty}^{\infty}\rho(s)\bar{y}_k^2(\tau)u^{-1}(\tau)d\tau\,,$$

in agreement with the special result for a square well found in Ref. [6]. The same sign of the rates for all the modes here may seem to contradict to the theorem (33). The contradiction is resolved, when short wavelength modes are taken into account. Namely, the wake function cannot be approximated by the Dirac function for so short waves, that their length is smaller than a scale of the wake function. These short waves introduce the required opposite sign contribution to the sum of the rates, making it zero.

Let's consider now an alternative case of a slowly decaying wake, that is we assume that $W(\tau)\approx -W_0=\mathrm{const}$. At small head-tail phases, the lowest mode has the same sign as the short-wake rate (34)

$$\mathrm{Im}Q_{w0}\cong 0.4\kappa N_b\zeta\sigma W_0\,;\quad \zeta\sigma\le 1$$

where σ is the rms bunch length and N_b is a number of particles in the bunch. The growth rates of the higher modes are of the opposite sign, making the rate sum equal to zero, Eq. (33). As a function of chromaticity, the growth rates reach their maxima at $\zeta\sigma \cong 0.7k$, where $\max(\mathrm{Im} Q_w) \cong 0.1\kappa N_b W_0$. After its maximum, the high order mode changes its sign at $\zeta\sigma \cong 0.7(k+1)$ to the same sign as the lowest mode, tending after that to

$$\mathrm{Im} Q_w \cong \kappa N_b W_0 /(4\zeta\sigma). \qquad (34)$$

All the numerical factors here were estimated using the data of Fig. 3, showing the coherent rates for the Gaussian bunch and constant wake function (see the next chapter), which are also not too far from the square well results of Fig.1.

MODES FOR GAUSSIAN BUNCH

The Gaussian distribution in phase space,

$$f(\mathrm{v},\tau) = \frac{N_b}{2\pi\sigma u} \exp(-\mathrm{v}^2/2u^2 - \tau^2/2\sigma^2), \qquad (35)$$

deserves a detailed consideration as a good example of solving the general problem ?, and due to its special practical importance. Indeed, this distribution function describes a thermal equilibrium of a bunch whose length is much shorter than the RF wavelength. It is convenient here to use natural units for this problem. The distance τ is measured in units of the bunch length σ, and the eigenvalue ν_k - in units of $u^2/\left(\sigma^2 Q_{\mathrm{eff}}(0)\right) = Q_s^2/Q_{\mathrm{eff}}(0)$. In these units, the boundary-value problem of Eqs. (25, 28) is written as

$$\nu_k \bar{y} + \frac{d}{d\tau}\left(e^{\tau^2/2}\frac{d\bar{y}}{d\tau}\right) = 0; \quad \bar{y}(\pm\infty) = 0. \qquad (36)$$

$$\int_{-\infty}^{\infty} \bar{y}_k(\tau)\bar{y}_m(\tau)d\tau = \delta_{km}$$

The orthogonality and normalization of the eigenfunctions here are consequences of self-ajointness of Eq. (36). This equation is easily solved numerically. The first four eigenfunctions are shown in Fig. 2.

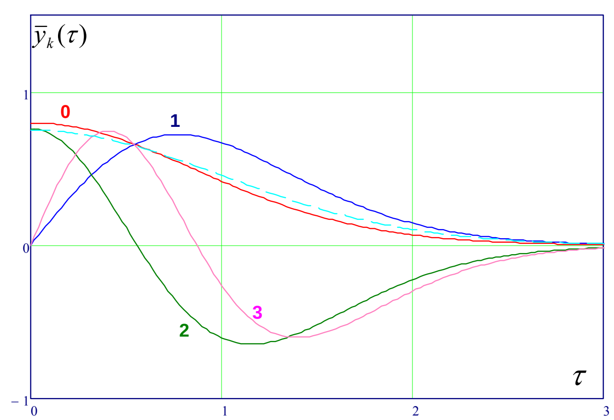

Fig. 2: The first four eigenfunctions for the Gaussian beam as functions of the dimensionless distance along the bunch τ, Eq. (36). The eigenfunctions are marked by their numbers. The dashed cyan line shows the bunch density, normalized similar to the eigenfunctions for comparison with the mode 0.

A list of first ten eigenvalues ν_k is found as follows:

Table 1: Eigenvalues of the Gaussian bunch

0	1	2	3	4	5	6	7	8	9
1.37	4.36	9.06	15.2	23.2	32.3	43.8	55.9	70.8	85.7

Each of these ten eigenvalues, except the first three, lies between two nearest squares of integers:

$$k^2 < \nu_k < (k+1)^2; \quad k = 3,4,5... \qquad (37)$$

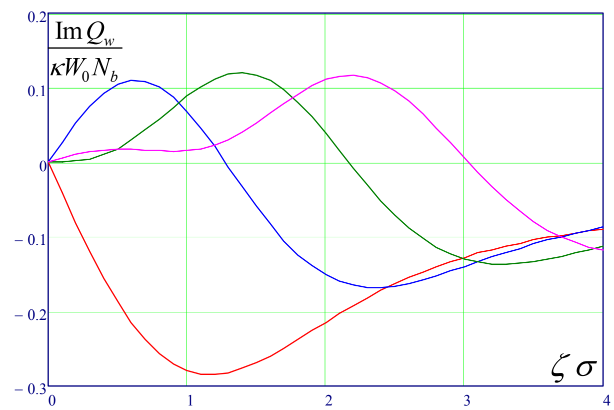

Fig. 3: Coherent growth rates for the Gaussian bunch with the constant wake as functions of the head-tail phase $\zeta\sigma$, for the lowest mode 0 (red), mode 1 (blue), 2 (green) and 3 (magenta). The rates are in units of $\kappa N_b W_0$, similar to the square well case of Fig. 1.

At their tails, $\tau \geq (1.5-2.5)$, the eigenfunctions follow the Gaussian asymptotic:

$$\begin{aligned} &\bar{y}_k(\tau) \sim C_k \exp(-\tau^2/2); \\ &C_0 = 0.5, C_1 = 0.85, \ C_2 = 1.4, \ ... \\ &\text{for } k \geq 2, \quad C_k \approx k/\sqrt{2} \end{aligned} \qquad (38)$$

In the next chapter, the mode energy numbers

$$E_k \equiv N_b^{-1}\int_{-\infty}^{\infty} \bar{y}_k^2(\tau)\rho(\tau)d\tau = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{y}_k^2(\tau)e^{-\tau^2/2}d\tau \qquad (39)$$

are needed. Calculation for the first ten modes shows extremely rapid convergence of that series:

$$\begin{aligned} &E_0 = 0.34, \ E_1 = 0.26, \\ &0.23 \leq E_k \leq 0.25, \quad \text{for } k \geq 2. \end{aligned} \qquad (40)$$

With the modes of the Gaussian bunch found, the coherent growth rates can be calculated according to Eq. (32), see Fig. 3. These growth rates of the Gaussian bunch look rather similar to the rates for the square well presented in Fig.1.

LANDAU DAMPING

Landau damping is a mechanism of dissipation of coherent motion due to transfer of the energy into incoherent motion. The coherent energy is effectively transferred only to resonant particles - the particles whose individual frequencies are equal to the coherent frequency. For these particles to exist, the incoherent spectrum must be continuous, not line. How can these conditions be satisfied for a bunched beam with strong space charge? It may seem at first glance, that when the space charge tune shift highly exceeds the synchrotron tune, the incoherent frequencies are so distant from the coherent line, that the resonant particles do not exist at all, making the Landau damping impossible. This conclusion though would not be correct. The reason is that it does not take into account that the space charge tune shift is not constant along the bunch: being maximal at the bunch center, it drops to zero at the tails. Thus, the Landau energy transfer is impossible at the bunch center, but it gets effective at the bunch tails, where the local incoherent space charge tune shift becomes small enough. Namely, it happens at the distance $\tau = \tau_*$ where a drop of the coherent amplitude felt by a moving particle is already so fast compared with the local incoherent tune shift, that this drop is not adiabatic. Then the energy of the coherent motion is effectively transferred to the incoherent motion. Thus, this position of the effective energy transfer occurs at the local space charge tune shift

$$Q_* \cong \mathrm{v}\partial / \partial\tau \,. \tag{41}$$

Here the gradient operator $\partial / \partial\tau$ has to be understood as applied to the dipole moment of the mode. For the Gaussian beam, the mode asymptotic Eq. (38) yields, in the same dimensionless units,

$$\partial / \partial\tau = 2\tau = 2\sqrt{2\ln(Q(0)/Q_*)} \,,$$

leading to:

$$Q_* = 2Q_s \mathrm{v}\sqrt{2\ln(Q(0)/Q_*)} \,. \tag{42}$$

Here $Q(\tau)$ is the individual space charge tune shift at position τ and given transverse actions.

$$Q(\tau) \equiv Q(J_1, J_2, \tau) \,.$$

The excited amplitude of the individual motion after passing this 'decoherence point' can be estimated as the mode amplitude there, $\bar{y}(\tau_*) \equiv y_*$. For the Gaussian bunch, according to Eq. (38),

$$y_* = C_k \exp(-\tau_*^2/2) = C_k Q_* / Q(0) \,. \tag{43}$$

Note that starting from this point $\tau \sim \tau_*$ and further to the tails, the particles do not already respond 'almost identically' to the coherent field, which was assumed from the very beginning in the rigid-beam model of Eq. (1). It means that any calculations based on properties of solutions of Eq. (25) at $\tau \approx \tau_*$ cannot have high accuracy, being only an estimation. Bearing this in mind, let us proceed with an estimation of the Landau damping, still keeping all the numerical factors. will not be too large. In other words, the damping rates are calculated below within the accuracy of a numerical factor, which, hopefully, should not be too large. A more accurate calculation will be left for future studies.

Before proceeding with the estimation though, the mentioned requirement for the continuous incoherent spectrum should be considered. Can this condition be satisfied without longitudinal or transverse non-linearity of the RF and the lattice? In the contrary to the no-space-charge case, it can, because the betatron phase advance per the synchrotron period depends both on the transverse and longitudinal incoherent amplitude, since the space charge tune shift is a function of these two amplitudes. For the Gaussian bunch, the space charge phase advance for a particle with the amplitude τ_0 per the synchrotron period $T_s = 2\pi/Q_s$ is calculated as

$$\begin{aligned}\Psi_s(\tau_0) &\equiv 4\int_0^{T_s/4} Q(\tau_0 \sin(Q_s\theta))d\theta = \\ \Psi_s(0)&\exp\left(-\frac{\tau_0^2}{4}\right) I_0\left(\frac{\tau_0^2}{4}\right) \sim \sqrt{\frac{2}{\pi}}\frac{\Psi_s(0)}{\tau_0}.\end{aligned} \tag{44}$$

This shows, that the individual spectrum of particles at the point τ_* is indeed continuous, and since

$$\Psi_s(0)/(2\pi) = Q(0)/Q_s >> 1 \,,$$

there are many lines of the resonant particles, numbered by integer n, for whom

$$\Psi_s(\tau_0)\big|_{\tau_0 \ge \tau_*} \equiv 2\pi n \,.$$

After M times of passing the decoherence point, the individual amplitude gets additionally excited by

$$\Delta y_i(M) = y_* \sum_{m=0}^{M-1} e^{im\psi} = y_* e^{iM\psi/2} \frac{\sin(M\psi/2)}{\sin(\psi/2)} .$$

Thus, the entire Landau energy transfer for the bunch after M >>1 turns can be expressed as

$$\Delta E(M) = 4\int d\mathbf{J} f(\mathbf{J}) y_*^2 \frac{\sin^2(M\psi/2)}{\sin^2(\psi/2)} \,,$$

where $\boldsymbol{J}$ is 3D vector of the three actions; leading and trailing bunch tails were taken into account. The contributions from particle entering and leaving the tails were assumed equal in magnitude but with random relative phase. From here, the power of the Landau energy transfer is calculated as

$$\begin{aligned}\Delta\dot{E} &= \frac{d\Delta E(M)}{T_s dM} = 4Q_s\int d\mathbf{J} f(\mathbf{J}) y_*^2 \delta_P(\psi), \\ \delta_P(\psi) &\equiv \sum_n \delta(\psi - 2\pi n).\end{aligned} \tag{45}$$

Here, we used the fact that at M >>1, $\sin(M\phi)/\phi = \pi\delta(\phi)$. Since the space charge phase advance Ψ is a big number, the sum over many resonance lines n can be approximated as an integral over these resonances, leading to

$$\Delta\dot{E} = \frac{2Q_s}{\pi}\left\langle \int_{J_*}^{\infty} dJ_\parallel f(J_\parallel) y_*^2 \right\rangle_\perp \,, \tag{46}$$

with $\langle\ldots\rangle_\perp$ meaning averaging over the transverse distribution, and $J_* = \tau_*^2/2$ is the longitudinal action for the decoherence point. For the Gaussian distribution, the longitudinal integral is calculated as

$$\int_{J_*}^{\infty} dJ_\parallel f(J_\parallel) = \exp(-\tau_*^2/2) = Q_*/Q(0).$$

After that, the Landau dissipation follows:

$$\Delta\dot{E} = C_k^2 \frac{2Q_s}{\pi}\left\langle \left(\frac{Q_*}{Q(0)}\right)^3 \right\rangle_\perp \,,$$

where the constants C_k are given by the asymptotic Eq. (38). According to Eq. (27), the transverse averaging in the last equation can be expressed in terms of the proper momentum of the tune shift, yielding

$$\Delta\dot{E} = C_k^2 \frac{2Q_s}{\pi}\left(\frac{Q_*}{q_{-3}Q_{\max}}\right)^3 ;\; q_{-3} = 0.52 .$$

This energy dissipation is directly related to the Landau damping rate Λ_k by $\Delta\dot{E} = 2\Lambda_k E_k$, with the energy number E_k given by Eq. (39,40). Eventually, this is leading to

$$\Lambda_k \cong 60 L^{3/2} \frac{C_k^2}{E_k}\frac{Q_s^4}{Q_{\max}^3} \;, \tag{47}$$

where $L \equiv \ln(0.5Q_{\max}/Q_*)$, and $Q_{\max} \equiv Q_{\max}(0)$ is the space charge tune shift in the 3D bunch center. For $k \geq 2$, $C_k^2/E_k \approx 2k^2$. The Landau damping rate (47) does not assume any nonlinearity of the external fields, neither longitudinal (RF), nor the transverse (nonlinear lattice elements). This makes a significant difference both with the standard no-space-charge head-tail theory, and the coasting beam Landau damping for arbitrary space charge, where the Landau damping is determined by external nonlinearities. Since the damping rate (47) is not related to these nonlinearities, it may be referred to as an immanent Landau damping.

Note that the synchrotron tune and the space charge tune shift enter in high powers in Eq. (47). It is worth noting , that contrary to a coasting beam case, the damping rate for the head-tail modes (47) does not depend on the chromaticity at all. However, the beam stability still strongly depends on the chromaticity due to the coherent tune shifts, Eq. (32). As it was mentioned above, the coherent growth rates $\mathrm{Im}(Q_w)$ are typically negative for the modes with numbers smaller than the head-tail phase, $k \leq \zeta\sigma$. If so, only high order modes, with $k \geq \zeta\sigma$ have to be stabilized by the Landau damping. For these modes, the Landau damping rate (47) goes as $\Lambda_k \propto k^2 \sim (\zeta\sigma)^2$. For the constant-like wakes, the maximal growth rate is achieved for a mode number $k \cong \zeta\sigma$, and it does not depend of the chromaticity. Thus, the stabilization parameter $\Lambda_k/\mathrm{Im}(Q_w)$ goes as $\Lambda_k/\mathrm{Im}(Q_w) \propto (\zeta\sigma)^2$ for the critical part of the coherent spectrum. Another possibility for stabilization is to keep zero chromaticity, where all the growth rates vanish. A possible problem for this method of stabilization is that the chromaticity has to be reliably kept rather close to zero.

LANDAU DAMPING BY LATTICE NONLINEARITY

In the previous chapter, the Landau damping was estimated for a linear lattice, where the bare tunes do not depend on the transverse amplitudes. If it is not so, the lattice tune spectrum is continuous, which may contribute an additional part to the entire Landau damping. This contribution is considered in this chapter.

To begin, let $\delta Q(J_2)$ be a nonlinear correction to the individual betatron tune for the 1st degree of freedom, which depends only on an alien action J_2, and does not depend of the own action J_1. Possible dependence on J_1 is taken into account later on. The nonlinearity modifies the single-particle equation of motion Eq. (4) as

$$\dot{y}_i(\theta) = iQ(\tau_i(\theta))[y_i(\theta) - \bar{y}(\theta,\tau_i(\theta))] - \delta Q_i y_i(\theta) \tag{48}$$

For positive non-linearity, $\delta Q_i > 0$, there is a certain point in the bunch, $\tau = \tau_r$, where the nonlinear tune shift exactly compensates the local space charge tune shift:

$$\delta Q_i = Q(\tau_r) \tag{49}$$

When the particle crosses this point, it actually crosses a resonance of its incoherent motion with the coherent one. Crossing the resonance excites the incoherent amplitude. Indeed, a solution of Eq. (48) is expressed similar to Eq. (20):

$$y_i(\theta) = -i\int_{-\infty}^{\theta} Q(\tau_i(\theta'))\bar{y}(\theta',\tau_i(\theta'))\exp(i\Psi(\theta) - i\Psi(\theta'))d\theta' ;$$
$$\Psi(\theta) \equiv \int_0^{\theta}[Q(\tau_i(\theta')) - \delta Q]d\theta' \quad . \tag{50}$$

Taking the integral (50) by the saddle-point method yields the incoherent amplitude excitation by the resonance crossing:

$$|\Delta y_i| = Q(\tau_r)\bar{y}(\tau_r)\sqrt{\frac{2\pi}{|\dot{Q}(\tau_r)|}} . \tag{51}$$

Here

$$\dot{Q}(\tau_r) = \frac{dQ}{d\tau}\frac{d\tau}{d\theta} = \frac{dQ}{d\tau}\mathrm{v}$$

is a time derivative of the local space charge tune shift in the resonance point (49) seen by the particle. For the Gaussian bunch, $\dot{Q} = -QQ_s\tau\mathrm{v}$, in the dimensionless units of length τ and velocity

$$\mathrm{v} = \sqrt{2(J_\parallel - J_r)} ;\quad J_r \equiv \tau_r^2/2 . \tag{52}$$

After the single-pass incoherent excitation is found, the multi-pass summation for the coherent energy dissipation can be done exactly as in the previous section, leading to an analogue of Eq. (46):

$$\Delta\dot{E} = \frac{2Q_s}{\pi}\left\langle \int_{J_r}^{\infty} dJ_\parallel f(J_\parallel)|\Delta y_i|^2 \right\rangle_\perp . \tag{53}$$

Like in the previous section, using that $\Delta\dot{E} = 2\Lambda_k E_k$, this energy dissipation gives the Landau damping rate

$$\Lambda_k = \frac{Q_s}{\pi E_k}\int d\mathbf{J}_\perp \int_{J_r}^{\infty} dJ_\parallel f(\mathbf{J}_\perp, J_\parallel)|\Delta y_i|^2 , \tag{54}$$

with $\mathbf{J}_\perp = (J_1, J_2)$ as 2D transverse action.

Up to this point, it was assumed that the nonlinear tune shift is independent of the action J_1 associated with the considered plane of the oscillation. If this dependence exists, the result has to be modified similar to conventional head-tail case (no-space charge), $f \to -J_1\partial f/\partial J_1$, leading to a general formula:

$$\Lambda_k = -\frac{Q_s}{\pi E_k}\int d\mathbf{J}_\perp \int_{J_r}^{\infty} dJ_\parallel \frac{\partial f(\mathbf{J}_\perp, J_\parallel)}{\partial J_1} J_1 |\Delta y_i|^2 . \tag{55}$$

Eq. (55) is valid for any distribution function and arbitrary nonlinearity.

For a Gaussian bunch (35), taking the longitudinal integral leads to

$$\Lambda_k = -\sqrt{\frac{\pi}{L}}\frac{C_k^2}{E_k}\int d\mathbf{J}_\perp J_1 \frac{\partial f_\perp}{\partial J_1}\frac{\delta Q^4}{Q^3(0)}\,, \qquad (56)$$

where $L = \ln\left(0.5 Q_{\max} / \langle \delta Q \rangle\right)$ and $\langle \delta Q \rangle$ is an average value for the nonlinear tune shift. The value for the 2D transverse integral Eq. (56) depends on the transverse distribution function $f_\perp(J_1, J_2)$ and the nonlinearity function $\delta Q(J_1, J_2)$. In general, with $Q(0) = Q_{\max} g(J_1, J_2)$, it can be presented as

$$\Lambda_k = \frac{A}{\sqrt{L}}\frac{C_k^2}{E_k}\frac{\langle \delta Q\rangle^4}{Q_{\max}^3}\,; \quad \langle \delta Q \rangle \equiv \int d\mathbf{J}_\perp f_\perp(J_1, J_2)\delta Q\,, \qquad (57)$$

where A is a numerical factor

$$A \equiv -\sqrt{\pi}\int d\mathbf{J}_\perp J_1 \frac{\partial f_\perp}{\partial J_1}\frac{1}{g^3(J_1, J_2)}\left(\frac{\delta Q}{\langle \delta Q\rangle}\right)^4 ,$$

For round Gaussian distribution Eq. (26, 26a) and symmetric octupole-driven nonlinear tune shift

$$\delta Q = \langle \delta Q \rangle (J_1 + J_2)/2\,, \qquad (58)$$

the transverse integration yields a big numerical factor:

$$A = 4.0 \cdot 10^3\,. \qquad (59)$$

A complete Landau damping rate is a sum of the two rates: the immanent rate found in the previous chapter and the nonlinearity-related rate of Eq. (57). Since the numerical factors for the two contributions are not too different, and their space charge dependence is the same, the leading contribution is determined by a ratio of the synchrotron tune and the average nonlinear tune shift.

VANISHING TMCI

When the wake-driven coherent tune shift is small compared with the distance between the modes, $Q_w << Q_s^2 / Q_{\max}$, it is sufficient to take it into account in the lowest order, as it is done by Eq. (32). When the wake force is not as small, that approximation is not justified. In this case, the modes are strongly perturbed by the wake forces; their shapes and frequencies significantly differ from the no-wake solutions of Eq. (25).

To complete the theory of the space charge modes, this equation has to be generalized for arbitrary wake field. To do so the wake term has to be dealt with in the same way as the space charge term, $\propto \bar{y}$ was dealt.

If the coherent tune shift is small compared with the space charge tune shift, the generalized equation for the eigen-modes follows:

$$\begin{gathered}
\nu\,\bar{y}(\tau) + u(\tau)\frac{d}{d\tau}\left(\frac{u(\tau)}{Q_{\text{eff}}(\tau)}\frac{d\bar{y}}{d\tau}\right) = \kappa\left(\hat{\mathbf{W}}\bar{y} + \hat{\mathbf{D}}\bar{y}\right) \\
\hat{\mathbf{W}}\bar{y} \equiv \int_\tau^\infty W(\tau - s)\exp(i\zeta(\tau - s))\rho(s)\bar{y}(s)ds\,; \\
\hat{\mathbf{D}}\bar{y} \equiv \bar{y}(\tau)\int_\tau^\infty D(\tau - s)\rho(s)ds\,.
\end{gathered} \qquad (60)$$

Note that this wake-modified equation is valid for any ratio between the coherent tune shift and the synchrotron tune; only small value of the coherent tune shift compared with the space charge tune shift, $Q_w << Q_{\max}$, is required.

A straightforward way to solve Eq. (60) is to expand the sought eigenfunction $\bar{y}(\tau)$ over the full orthonormal basis of the no-wake modes $\bar{y}_{0k}(\tau)$:

$$\bar{y}(\tau) = \sum_{k=0}^{\infty} B_k \bar{y}_{0k}(\tau)\,, \qquad (61)$$

with as yet unknown amplitudes B_k. After that, the integro-differential Eq. (60) is transformed into a linear matrix problem for eigen-solutions

$$(\kappa\hat{\mathbf{W}} + \kappa\hat{\mathbf{D}} + \text{Diag}(\mathbf{v}_0))\mathbf{B} = \nu\,\mathbf{B}\,. \qquad (62)$$

Here $\hat{\mathbf{W}}$ and $\hat{\mathbf{D}}$ are the matrices of the driving and detuning wake operators in the basis of the no-wake modes:

$$\begin{gathered}
\hat{\mathbf{W}}_{km} = \int_{-\infty}^{\infty}\int_\tau^{\infty} W(\tau - s)\exp(i\zeta(\tau - s))\rho(s)\bar{y}_{0k}(\tau)\bar{y}_{0m}(s)u^{-1}(\tau)ds d\tau\,, \\
\hat{\mathbf{D}}_{km} = \int_{-\infty}^{\infty}\int_\tau^{\infty} D(\tau - s)\rho(s)\bar{y}_{0k}(\tau)\bar{y}_{0m}(\tau)u^{-1}(\tau)ds d\tau\,;
\end{gathered} \qquad (63)$$

a symbol $\text{Diag}(\mathbf{v}_0)$ represents a diagonal matrix whose k-th diagonal element is a k-th eigenvalue of the no-wake problem ν_{0k}:

$$\text{Diag}(\mathbf{v}_0)_{mn} = \nu_{0m}\delta_{mn}\,;$$

and $\mathbf{B}$ is a vector of the mode amplitudes in Eq. (61).

In the conventional no-space-charge head-tail theory (see e. g. Ref.[4]), the beam is stable at zero chromaticity and the wake amplitude below a certain threshold. When the wake term grows, it normally moves the tunes $\nu_k = kQ_s$, $k = 0, \pm 1, \pm 2, \ldots$ increasingly down. Mostly the tune of the mode number 0, or the base tune, is moved. As a result, that base tune meets at some threshold intensity the nearest from below tune of the mode -1, which is typically moved not as much. Starting from this point, the transverse mode coupling instability (TMCI) occurs. The threshold value of the coherent tune shift is normally about the synchrotron tune.

There is a significant structural difference between the conventional synchro-betatron modes and the space charge modes introduced here. Namely, the conventional modes are numbered by integer numbers, both positive and negative, while the space charge modes are numbered by natural numbers only. All the tunes of the space charge modes are positive, increasing quadratically with the mode number. Due to fundamental properties of wake functions, the tunes are normally moved down by the wake force, and normally the mode 0 is mostly affected. At this point an important difference between the conventional and the space charge modes appears. Namely, in the conventional case, the 0 mode has a neighbor with lower tune. On the contrary, the space charge lowest mode does not have a neighbor from below; thus, its shifted downward tune cannot cross a tune of some other mode. Moreover, since the wake-driven tune shift normally decreases with the mode number, the wake field works as a factor of divergence of the coherent tunes. As soon as this typical picture is valid, TMCI is impossible. An illustration of this mode behavior is presented in Fig.4, where coherent tunes of the Gaussian bunch are shown as functions of the wake amplitude for a case of a constant wake $W = -W_0 = \mathrm{const} < 0$, no detuning, and zero chromaticity. The eigenvalues are given in the same units as they were calculated in the no-wake case and presented in Table 1. The TMCI still appears, as it is seen in Fig.4, but at very high values of the wake field, where formally calculated lowest order coherent tune shift is an order of magnitude higher than a tune separation between the lowest 2 modes. A similar mode behavior is shown in Fig. 5, for the square well model with a resistive wall wake function, $W(\tau) = -W_0 / \sqrt{\tau}$.

The conclusion about vanishing TMCI is also confirmed by calculations of Ref. [6] for an air-bag bunch in the square well with an exponential wake $W(-\tau) = -W_0 \exp(-\alpha\tau)$. In particular, Fig.14 of that article shows disappearance of TMCI for any wake length 1/α, as soon as the wake-driven coherent tune shift is exceeded by the space charge tune shift. Conclusion of this TMCI suppression at strong space charge should not be taken though for granted for any wake function. TMCI suppression should not be expected for significantly oscillating wake functions, for which diagonal matrix elements have a maximum for one of the high-order modes.

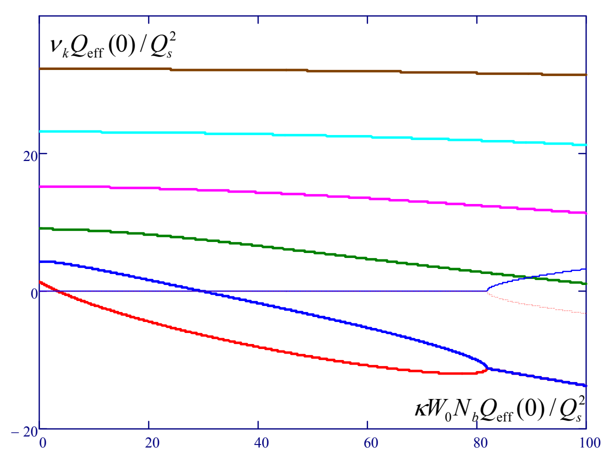

Fig.4: Coherent tunes of the Gaussian bunch for zero chromaticity, constant wake and no detuning, versus the wake amplitude. The eigenvalues are in the units of Table 1, $Q_s^2 / Q_{\mathrm{eff}}(0)$; the wake amplitude on the horizontal axis is presented as $\kappa N_b W_0 Q_{\mathrm{eff}}(0) / Q_s^2$. Thick lines show the real parts of the eigenvalues. The thin red and blue lines show the growth rates appearing after coupling of the lowest 2 modes. Note high value of the TMCI threshold. Need to label the lines.

As a result, mode coupling would appear at rather low coherent tune shift for the oscillating wake functions. A mode behavior like that is illustrated in Fig. 6, where the coherent frequencies are calculated for the square potential well with an oscillating wake function $W(\tau) = -W_0 \cos(2\pi\tau / l)$. As it is seen, the 1-st mode is the most affected by the wake field, and the TMCI threshold is close to a point where it can be expected from the zero-wake slope of the 1-st mode; there is no any suppression of TMCI.

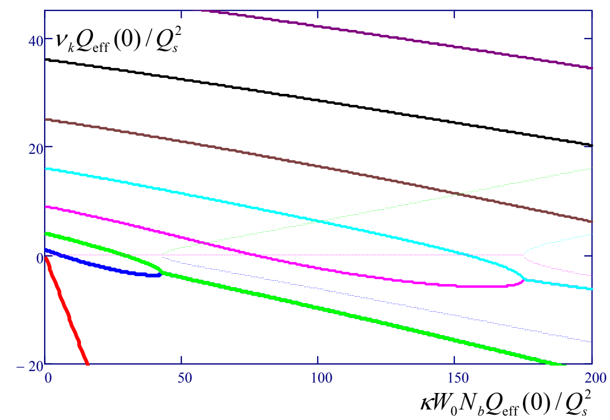

Fig.5: Same as Fig.4, but for the square well and resistive wake function $W(\tau) = -W_0 \sqrt{l / \tau}$. Note significantly postponed TMCI threshold. Thin lines show growth rates; colors are matched.

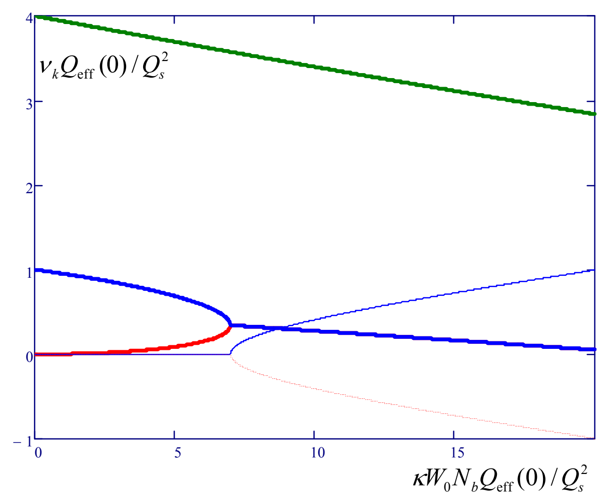

Fig. 6: Same as Fig. 5, but for an oscillating wake function $W(\tau) = -W_0 \cos(2\pi\tau / l)$. Note that mode coupling occurs where it can be expected from low-wake behavior of the most affected mode.

From practical point of view, wakes of hadron machines are typically dominated by the resistive wall contributions, which is constant-like in that sense. As soon as it is so?, the TMCI threshold is significantly increased when the space charge tune shift exceeds the synchrotron tune.

To summarize, an entire picture of the TMCI threshold can be described for arbitrary ratio of the space charge tune shift and the synchrotron tune. When this ratio is small, the conventional TMCI theory is applicable, giving approximately the synchrotron tune as the threshold value for the maximal coherent tune shift. When the space charge tune shift starts to exceed the synchrotron tune, the TMCI threshold for the coherent tune shift $(Q_w)_{\rm th}$ is approximately determined by a minimum of two values: the space charge tune shift $Q_{\max}$ and the lowest tune of the space charge mode $\sim Q_s^2/Q_{\max}$, multiplied for non-oscillating wakes by a rather big numerical factor, 20-100. A very schematic picture of the TMCI threshold as a function of the space charge tune shift over the synchrotron tune is presented in Fig.7.

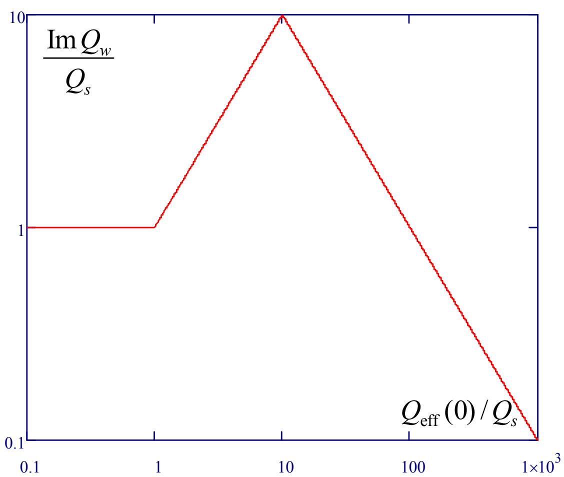

Fig.7: A schematic behavior of the TMCI threshold for the coherent tune shift versus the space charge tune shift. Both tune shifts are in units of the synchrotron tune.

This schematic picture would be additionally modified by multi-turn wake field or multiple bunches; this issue though is outside the scope of this paper.

SUMMARY

In this paper, a theory of head-tail modes is presented for space charge tune shift significantly exceeding the synchrotron tune, which is rather typical case for hadron machines. A general equation for the modes is derived for any ratio of the synchrotron tune and the wake-related coherent tune shift. Without the wake term, this is a 2-nd order self-adjoint ordinary differential equation with zero boundary conditions, known to have full orthonormal basis of the eigenfunctions. The spectrum of this equation is discussed in general and solutions for the Gaussian bunch are presented in detail. Landau damping of the space charge modes is considered and calculated both without and with lattice nonlinearity. It is found that the Landau damping rates are inversely proportional to the space charge tune shift cubed, and proportional to the fourth power of a maximum from the synchrotron tune and the average lattice nonlinear tune shift. Finally, the transverse mode coupling instability for the space charge modes is developed. It is shown, that typically the TMCI threshold is 1-2 orders of magnitude higher than that naively expected from the small wake behavior of the lowest mode.

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